Fell's absorption principle for semigroup operator algebras

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The statement of the problem

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The obvious choice does not work!

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One way to identify the desired universal object is to isolate properties of the left regular representation and ask that our generic representation satisfies them.

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More generally, $L^{\otimes n}$ and $L \otimes \pi$, for any isometric representation π of P.

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Theorem (Fell's absorption principle)

If π is any unitary representation of G then $L \otimes \pi$ and $L \otimes \mathrm{id}$ are unitary equivalent.

Li's universal C^*-algebra

Let P be a semigroup of a group G. For each $k \in \mathbb{N}$ we consider the set of words of length $2k$ in P,

$$\mathcal{W}(P)^k := \{(p_1, p_2, \ldots, p_{2k-1}, p_{2k}) \mid p_j \in P, \text{ for } j = 1, 2, \ldots, 2k\}$$

and we let $\mathcal{W}(P) := \cup_{k=0}^{\infty} \mathcal{W}(P)^k$, with the understanding $W(P)^0 = \emptyset$. (When the context makes it clear what P is, we simply write $\mathcal{W}$ instead of $\mathcal{W}(P)$.)

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A word $a \in \mathcal{W}$ is said to be neutral if $\dot{a} = e$, the neutral element of G. If $V = \{V_p\}_{p \in P}$ is an isometric representation of P and $a = (p_1, p_2, \ldots, p_{2k-1}, p_{2k}) \in \mathcal{W}$, then we define

$$\dot{V}_a := V_{p_1}^* V_{p_2} V_{p_3}^* \cdots V_{p_{2k-1}}^* V_{p_{2k}}.$$

If $a = (p_1, p_2, \ldots p_{2k-1}, p_{2k})$ is a word in $\mathcal{W}(P)$, we write

$$K(a) := P \cap (p_{2k}^{-1} p_{2k-1})P \cap (p_{2k}^{-1} p_{2k-1} p_{2k-2}^{-1} p_{2k-3})P \cap \cdots \cap (\dot{\tilde{a}})P,$$

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We let

$$\mathfrak{J}(P) := \{K(a) \mid a \in \mathcal{W}(P)\},$$

dropping the reference to P and simply writing $\mathfrak{J}$, if there is no source of confusion.

Definition

Let P be a submonoid of a group G and let $\mathfrak{J}$ be the collection of all constructible right ideals of P. Li's semigroup C^*-algebra of P, denoted as $\mathrm{C}^*_s(P)$, is the universal C^*-algebra generated by a family of isometries $\{w_p\}_{p \in P}$ such that

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In particular, any map $w : P \to B(\mathcal{H})$ satisfying the relations (T1), (T2) and (T3) is a representation of P by isometries.

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If in addition there is a conditional expectation $E : A \to \mathcal{B}_e$ which vanishes on $\mathcal{B}_g$ for $g \neq e$, we say that the pair $(\{\mathcal{B}_g\}_{g \in G}, E)$ is a *topological* grading of A.

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With the use of representations, we associate with $\mathcal{B}$ two cross-sectional C^*-algebras: a maximal one $\mathrm{C}^*(\mathcal{B})$ and a minimal one $\mathrm{C}^*_r(\mathcal{B})$ for representations generating a topological grading.

Theorem (Exel '97)

Let $\mathcal{B} = \{\mathcal{B}_g\}_{g \in G}$ *be a topological grading and let* $\pi = \{\pi_g\}_{g \in G}$ *be a representation of* $\mathcal{B}$. *If* $l : G \to B(\ell^2(G))$ *is the left regular representation of* G, *then* $\mathrm{C}^*(\pi \otimes l)$ *is canonically isomorphic to* $\mathrm{C}^*_r(\mathcal{B})$.

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Theorem

Let $\mathcal{B} = \{\mathcal{B}_g\}_{g \in G}$ be a topological grading and let $\pi = \{\pi_g\}_{g \in G}$ be a representation of $\mathcal{B}$ generating a topological grading with associated conditional expectation $F : \mathrm{C}^*(\pi) \to \pi_e(B_e)$. Then there exists a canonical $*$-epimorphism

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satisfying

$$\ker\lambda = \{x \in \mathrm{C}^*(\pi) \mid F(x^*x) = 0\}.$$

Proposition

Let P be a submonoid of a group G. Then there is a faithful $*$-representation (coaction)

$$\delta\colon \mathcal{T}_\lambda(P) \longrightarrow \mathcal{T}_\lambda(P) \otimes \mathrm{C}^*(G); L_p \longmapsto L_p \otimes u_p.$$

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A similar grading exists for $\mathrm{C}^*_s(P)$.

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Consider the additive semigroup $\mathbb{N}_* := \mathbb{N} \backslash \{1\}$. We show that $\mathbb{N}_*$ does not satisfy independence. Indeed, consider the constructible ideals

$$k + \mathbb{N}_* = L_k L_k^* = \{k, k+2, k+3, \ldots\}, \; k = 2, 3, \ldots.$$

Recall that a cancellative semigroup P is said to satisfy independence if for every $X \in \mathcal{J}$ and all $X_1, X_2, \ldots, X_n \in \mathcal{J}$,

$$X = \cup_{i=1}^{n} X_i$$

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$5 + \mathbb{N} = (5 + \mathbb{N}_*) \cup (6 + \mathbb{N}_*)$,and yet $5 + \mathbb{N} \neq k + \mathbb{N}_*$, for any $k = 2, 3, \dots$. Hence $\mathbb{N}_*$ does not satisfy independence.

Proposition

Let $P \subseteq G$ be a submonoid that does not satisfy independence. Then the map

$$L_p \longmapsto L_p \otimes L_p, p \in P$$

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Question 1: Identify relations for which $\mathrm{C}^*(\{\mathcal{T}_\lambda(P)\}_g)$ is the universal algebra.
Question 2: Identify $\mathrm{C}^*_r(\{\mathrm{C}^*_s(P)\}_g)$.
Question 3: How does Fell's absorption principle manifests for arbitrary submonoids?

Answers

Question 1 was answered recently by Laca and Sehnem.

Theorem (Laca and Sehnem 2021)

The C^**-algebra* $\mathrm{C}^*(\{\mathcal{T}_\lambda(P)\}_g)$ *is the universal algebra for the relations*

(T1) $w_e = 1$;

(T2) $\dot{w}_a = 0$, *if* $K(a) = \emptyset$ *with* $\dot{a} = e$;

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(T4) $\prod_{b \in F}(\dot{w}_a - \dot{w}_b) = 0$, *if* F *is a finite set of neutral words with* $K(a) = \cup_{b \in F} K(b)$, *for some neutral word* a.

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$$\bar{L} : P \longrightarrow B\left(\oplus_{n=1}^{\infty} \ell^2(P)^{\otimes n}\right); \ P \ni p \longmapsto \oplus_{n=1}^{\infty} L_p^{\otimes n},$$

where $L = \{L_p\}_{p \in P}$ denotes the left regular representation of P. (For notational simplicity, we will be writing $\bar{\ell}^2(P)$ instead of $\oplus_{n=1}^{\infty} \ell^2(P)^{\otimes n}$.)

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The generated C*-algebra (resp. operator algebra) is denoted by $\bar{\mathcal{T}}_\lambda(P)$ (resp. $\bar{\mathcal{T}}_\lambda^+(P)$).

Proposition

Let P be a submonoid of a group G. Then there is a coaction

$$\bar{\delta}\colon \bar{\mathcal{T}}_\lambda(P) \longrightarrow \bar{\mathcal{T}}_\lambda(P) \otimes \mathrm{C}^*(G); \bar{L}_p \longmapsto \bar{L}_p \otimes u_p.$$

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Moreover the spectral subspaces

$$\bar{\mathcal{T}}_\lambda(P)_g = \overline{\mathrm{span}}\{\dot{\bar{L}}_a \mid a \in \mathcal{W}, \dot{a} = g\}.$$

form a topological grading for $\bar{\mathcal{T}}_\lambda(P)$.

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Corollary

Let P be a submonoid of an abelian group G, e.g., $\mathbb{N}_*$. Then Li's semigroup C^*-algebra $\mathrm{C}^*_s(P)$ is canonically isomorphic with $\bar{\mathcal{T}}_\lambda(P)$, i.e., the C^*-algebra generated by the enhanced left regular representation.

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We always have (i) $\iff$ (ii) $\implies$ (iii).

Theorem (K. 2023)

Let P be a submonoid of a group G. Consider

(i) $\mathrm{C}_s^*(P)$ *is nuclear*

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We always have (i) $\iff$ (ii) $\implies$ (iii). *If G is exact, then all of the above conditions are equivalent.*

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Theorem (Fell's absorption principle for semigroups, K. 2023)

Let P be a submonoid of a group G. Let $\bar{L}$ be the enhanced left regular representation of P and let π be a $*$-representation of $\mathrm{C}_s^*(P)$. Then the map

$$\bar{\mathcal{T}}_\lambda(P) \ni \bar{L}_p \longmapsto \bar{L}_p \otimes \pi(w_p), \; p \in P, \tag{1}$$

extends to an injective representation of $\bar{\mathcal{T}}_\lambda(P)$.

The non-selfadjoint theory

Theorem (K. 2023)

Let P be a submonoid of a group G and let $\bar{\mathcal{T}}_\lambda(P)^+$ denote the non-selfadjoint algebra generated by the enhanced left regular representations $\bar{L}$. Then $\bar{\mathcal{T}}_\lambda(P)^+$ is completely isometrically isomorphic to $\mathcal{T}_\lambda(P)^+$ via a map that sends generators to generators.

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and note that the completely contractive map defined by

$$\mathcal{T}_\lambda(P)^+ \ni L_p \overset{\delta}{\longmapsto} L_p \otimes u_p \longmapsto L_p \otimes (l_p|_{\ell^2(P)}) = L_p \otimes L_p$$

is multiplicative because $\ell^2(P)$ is invariant by all l_p, $p \in P$.

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Iterations of the above argument show that the maps

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are completely contractive and multiplicative. By taking a direct sum of all these maps, we produce a completely isometric map from $\mathcal{T}_\lambda(P)^+$ onto $\bar{\mathcal{T}}_\lambda(P)^+$, which sends generators to generators.

Corollary

Let P be a submonoid of a group G. Then there is a completely isometric comultiplication Δ_P on $\mathcal{T}_\lambda(P)^+$ given by $\Delta_P(L_p) = L_p \otimes L_p$, $p \in P$.

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Then the map $\Delta_P := (\psi^{-1} \otimes \psi^{-1})\phi\psi$ is the desired comultiplication. ∎

Applications

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This last result works for semigroups that do not satisfy independence and therefore allows us to strengthen recent results of Clouatre and Dor-On.

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Corollary

Let P be a submonoid of an amenable group G. Assume that P is left reversible and has FDP, e.g., $\mathbb{N}_*$.

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Let P be a submonoid of an amenable group G. Assume that P is left reversible and has FDP, e.g., $\mathbb{N}_*$. Then $\mathrm{C}^*_{\max}(\mathcal{T}_\lambda(P)^+)$ is RFD.

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Clouatre and Dor-On established the previous result under the assumption that P satisfies independence. Other results are included in our paper.